On Numerical Approximation of the DMC Channel Capacity

(BFA'2017 Workshop)

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(5^{th} July, 2017)

Outline

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  1) the sample number is denoted by S,
  2) the dimension of the signal source is denoted by 2^n,
  3) the Walsh spectrum of the source distribution is denoted by the three valued set $\{0, +d, -d\}$, where the value d and the number k of nonzero coefficients are unknown variables.

Walsh Spectrum Characterization on Sampling Distributions (cont'd)

- Given an input array $\mathbf{x} = (x_0, x_1, \ldots, x_{2^n-1})$ of 2^n reals in the time domain, the Walsh transform $\mathbf{y} = \widehat{\mathbf{x}} = (y_0, y_1, \ldots, y_{2^n-1})$ of $\mathbf{x}$ is

$$y_i \stackrel{\text{def}}{=} \sum_{j \in GF(2)^n} (-1)^{\langle i,j \rangle} x_j, \text{ for } i \in GF(2)^n.$$

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- The main problem asks to obtain *as precise and much knowledge as possible* about the signal source D from the sampling distribution D' using S samples.
- The main goal is to find out some large or even the largest nontrivial Walsh coefficient(s) and the index position(s) for D.

Important Comments

- This work is the follow-up result of [Lu-Desmedt'2016], [Lu'2016] and has origins in linear cryptanalysis (cf. [Lu-Vaudenay'2008], [Molland-Helleseth'2004]).

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  1) the dimension 2^n is very large (e.g., 2^{64}),
  2) Walsh spectrum is not just a three valued set,
  3) D is an un-normalized distribution.
- The proposed problem incorporates the case that the source distribution D has zeros in the time domain.

Outline

Background

Channel Capacity Calculation

Further Discussions

Conclusion

Motivation on Studying Channel Capacity

- Inspired by the idea of compressive sensing, [Lu'2015] first constructed *imaginary channel transition matrices* $T \stackrel{\text{def}}{=} p(y|x)$ of size 2×2 and $2 \times M$, and introduced Shannon's channel coding problem to statistical cryptanalysis.

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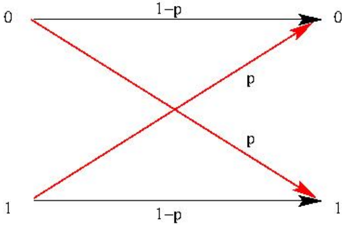

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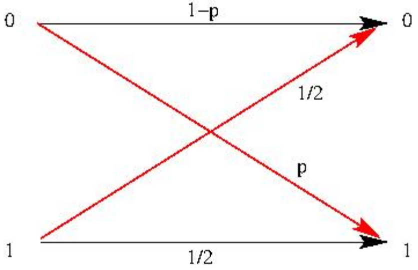

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- Case Three: Non-Binary Non-square Channel

$$T = \begin{bmatrix} D \\ U \end{bmatrix},$$

D, U denote the source distribution and the uniform distribution over the binary vector space of dimension n respectively.

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- In above Case Three, $C(T)$ gives a perfect answer to the key question in cryptanalysis: What is the minimum number of data samples to distinguish one biased distribution from the uniform distribution?

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- Note that the most recent work [Sutter et al'2014] has the complexity $O(M^2 N \sqrt{\log N}/\epsilon)$ for the same problem.

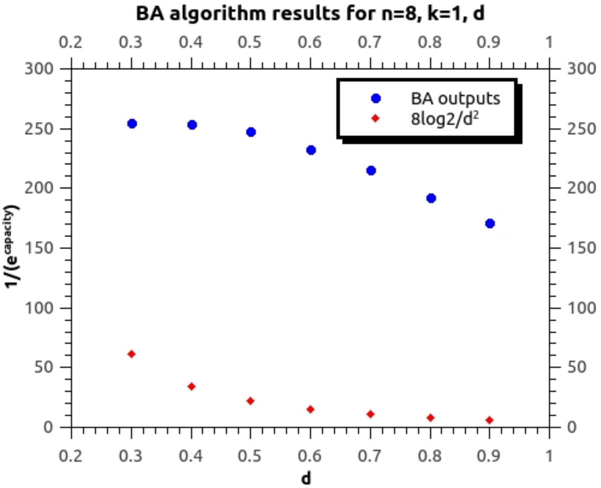
Blahut-Arimoto Algorithm in Pseudo-Codes

Input:

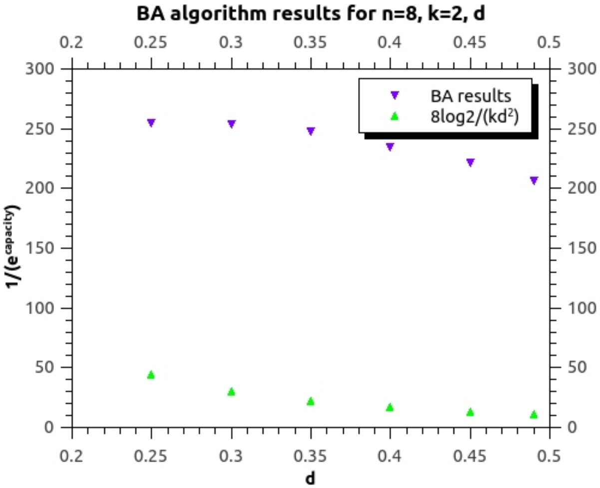
$Q_{k|j}$: transition matrix of size 2×2^n
(p_0, p_1): input distribution vector
ϵ : the desired absolute accuracy

initialize the values of $Q_{k|j}$ and p_0, p_1
repeat
  $c_0 \leftarrow exp\left(\sum_{k=0}^{2^n-1} Q_{k|0} \log \frac{Q_{k|0}}{p_0 Q_{k|0} + p_1 Q_{k|1}}\right)$
  $c_1 \leftarrow exp\left(\sum_{k=0}^{2^n-1} Q_{k|1} \log \frac{Q_{k|1}}{p_0 Q_{k|0} + p_1 Q_{k|1}}\right)$
  $I_L \leftarrow \log(p_0 c_0 + p_1 c_1)$
  $I_U \leftarrow \log \max(c_0, c_1)$
  update p_0 by $p_0 c_0 / (p_0 c_0 + p_1 c_1)$
  update p_1 by $p_1 c_1 / (p_0 c_0 + p_1 c_1)$
until $|I_U - I_L| < \epsilon$
output I_L

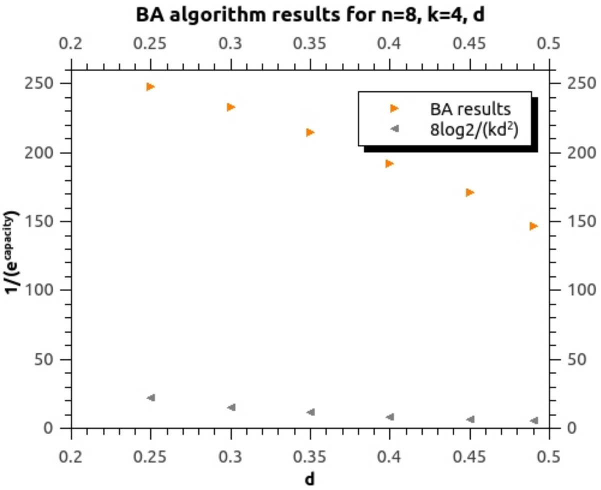
Capacity Results for $n = 8, k = 1$



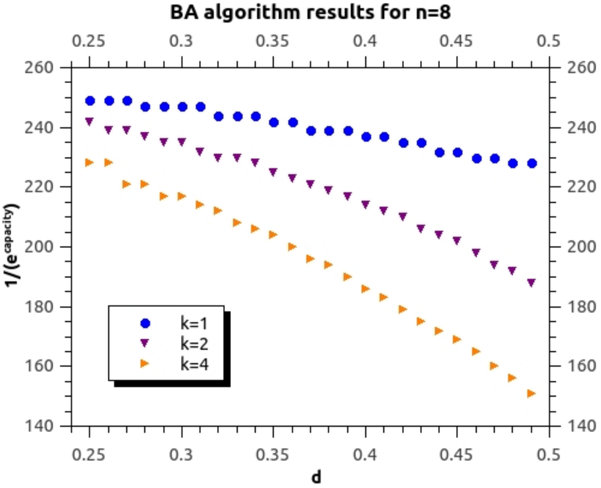
Capacity Results for $n = 8, k = 2$ (cont'd)



Capacity Results for $n = 8, k = 4$ (cont'd)



Capacity Results for $n = 8, \epsilon = 0.01$ (cont'd)



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About High-Precision Numerical Computation Software

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- In the new era of big data, high-precision numerical computation software is badly needed.
- Current available software and libraries with the feature:
  - MATHEMATICA
  - MATLAB
  - GNU Multiple Precision Arithmetic Library (GMP)
  - GNU Scientific Library (GSL)
  - etc.

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- Check the value of $c_0 = exp(TMP1 - TMP2)$:

$$TMP1 = \frac{3}{8}\log(\frac{3}{1024}) + \frac{5}{8}\log(\frac{5}{1024}) \quad (1)$$

$$TMP2 = \frac{42 \times 0.8}{8 \times 1024} = \frac{4.2}{2^{10}} \quad (2)$$

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To finalize,

- check the value of I_U:

$$\begin{aligned} \log c_0 &= TMP1 - TMP2 = -5.51\underline{3} \\ I_U &= \max(-5.51\underline{3}, -5.54\underline{9}) = -5.51\underline{3} \end{aligned}$$

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  as $\log(\cdot)$ and $exp(\cdot)$ both increase with the input.
- As $|I_U - I_L| < 0.1$, we now know $I_L = -5.5\underline{X}$.
- Meanwhile, the computer running BA algorithm also outputs I_L: "-5.5", i.e., to be interpreted as $]-5.5-0.1, -5.5+0.1[$.

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- For lower value of ϵ and $k > 1$, manual checking becomes harder for (1-3).
- Open Question:
  Evaluate the output precision of a composite function, which has exact values of inputs *initially*.

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- Our implementation allows to solve a lower-bound for distinguishing two distributions with arbitrarily small error probability.
- We have done experiments in the setting of Sparse Walsh Spectrum with $M = 2^8, \epsilon = 0.01, k = 1, 2, 4$ and one distribution is a uniform distribution.

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- We have examined the important issue of calculation precision with $M = 2^8, \epsilon = 0.1, k = 1$.
- We are carrying out challenging large-scale experiments with larger M and more values of k.

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